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Finite Element–Based Boundary Treatment in the Hybrid Particle Method

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I. Introduction

THE hybrid particle method (HPM) is a recently developed particle based method for the solution of high-speed dynamic structural problems.[1,2] Increased stability[3] and accuracy[1] of the HPM over colocational particle methods, such as the classical smooth particle hydrodynamics,[4] has been demonstrated.

The HPM is based on the strong form of the conservation equations. In general, particle methods based on such equations and that employ moving-least-squares (MLS) interpolants face difficulties in finding accurate derivatives of stress components at natural boundaries, such as stress-free surfaces and corners. This is because of the presence of asymmetric or skewed neighborhoods at natural boundaries. To increase precision, special numerical formulations are required to impose proper stress-free surface conditions.[2] These involve the tracking of the direction of the surface normal, which not only increases the computational cost of the analysis but can also prove to be cumbersome for three-dimensional analyses. Although these tentative procedures for boundary treatment do provide adequate results, it is of interest to derive more efficient procedures that lend themselves to straightforward implementation.

The explicit finite element method (FEM) is based on the weak form of the conservation equations. It does not require the explicit calculation of stress derivatives on natural boundaries. Thus, the difficulties associated with the treatment of boundaries using the HPM are not present in explicit FEM. We propose an approach in which the motion points of the HPM that lie on natural boundaries are treated based on the explicit FEM. The formulation of this approach is simple and does not require either the tracking of the surface normal or the explicit enforcement of the stress-free surface conditions.

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II. HPM Formulation

The HPM uses as its basis the differential form of equations for the conservation of momentum, mass, and energy, respectively. The formulation employs two types of points to facilitate computations. These are the motion points (mps) and the stress points (sps), respectively. The motion points are locations where the momentum is balanced and where the accelerations, velocities, and displacements are computed directly using the central difference method. The stress points are locations where velocities are obtained explicitly through direct interpolation from the surrounding motion point neighborhood.[1] Stress histories are tracked at both the motion points and the neighboring stress points.

The solution algorithm for the governing equations within a time step including contact treatment is described elsewhere.[2] A brief review of the solution procedure within the time step (t^n, t^{n+1}) is as follows:

1) Compute the acceleration $\boldsymbol{a}$ at each motion point at time t^n by the direct use of the momentum conservation relation. The derivatives of the stress components in this relation are computed through MLS interpolations over surrounding stress-point neighbors.

2) Given the acceleration $\boldsymbol{a}$, compute both the velocity $\boldsymbol{v}$ at $t^{n+1/2}$, where $t^{n+1/2} = (t^n + t^{n+1})/2$, and the position vector $\boldsymbol{x}$ at time t^{n+1} for each motion point using the central difference method.

3) Compute the velocity $\boldsymbol{v}$ for each stress point at time $t^{n+1/2}$ by interpolation from velocities of neighboring motion points.

4) For each mp and sp, a) find the specific volume V by using the difference form of the conservation of mass; b) compute the pressure p using either the equations of state (EOS) or bulk Hooke's law, and calculate the deviatoric stress $\boldsymbol{S}$ using the appropriate constitutive relations; c) compute the stress tensor from the pressure and deviatoric stress in b; and d update energy and artificial viscosity.

5) Calculate the next time step using the Courant condition, set $t = t^{n+1}$, and return to step 1.

III. FE-Based Motion Points

Step 1 of the HPM formulation requires the use of MLS interpolants to compute the derivatives of certain stress components at each motion point. For motion points located on natural boundaries, such as the stress-free surfaces or corners, the use of MLS interpolants in computing these derivatives leads to a lack of accuracy because skewed stress-point neighborhoods exist for these motion points.[1] For three-dimensional problems, skewed-stress point neighborhoods need to be dealt with for motion points on stress-free surfaces, stress-free edges (intersection of two stress-free surfaces), and stress-free corners (intersection of three or more stress-free surfaces).

Explicit FEM uses stresses at integration points and the shape functions of the surrounding elements to compute the acceleration at an element node and does not require the explicit calculation of stress derivatives. We propose that the pseudocells in the HPM be viewed as finite elements with the motion points corresponding to the pseudocells forming the nodes of these elements. The stress points that lie within a pseudocell can serve as the integration points for the finite elements. This concept is illustrated for two dimensions in Figs. 1a and 1b. The relations developed for calculating accelerations at nodes using explicit FEM can now directly be used to calculate accelerations at motion points instead of using the MLS interpolants. The motion points whose accelerations are

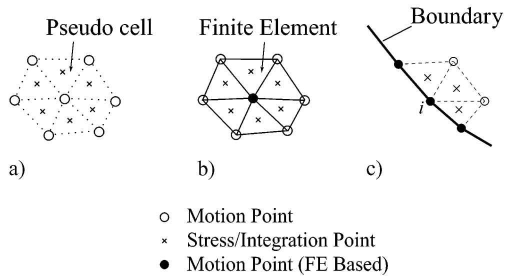

Fig. 1 Illustration of the concept of the FE-based motion points: a) motion point and its stress-point neighbors; each stress point is located at the center of the pseudo cell; b) viewing pseudocells as finite elements; the acceleration of the FE-based motion point can be computed by using explicit FEM relations; and c) use of FE-based motion points on the stress-free boundary.

calculated using explicit FEM are referred to here as the FE-based motion points. The special treatments proposed in the literature for the HPM to cope with the asymmetrical neighborhoods are no longer needed in cases where FE-based motion points are employed.

A. Accelerations at Element Nodes by Using Explicit FEM

By using the customary FE-based shape functions, the velocity and the variation in displacement at $\boldsymbol{x}$ can be expressed, respectively, as

$$\boldsymbol{v} = N_\alpha(\boldsymbol{x})\boldsymbol{v}_\alpha, \qquad \delta\boldsymbol{x} = N_\alpha(\boldsymbol{x})\delta\boldsymbol{x}_\alpha, \qquad \text{summation on} \quad \alpha \tag{1}$$

where $\alpha = 1, N^E$; N^E is the number of nodes per element; N_α, $\boldsymbol{v}_\alpha$, and $\boldsymbol{x}_\alpha$ are the shape function, the velocity vector, and the position of the node α, respectively; and δ denotes variation. Substituting the preceding relations into the weak form of momentum conservation and employing a lumped mass approach[5] leads to

$$\dot{\boldsymbol{v}}_\alpha = -\frac{1}{M_\alpha}\int_\Omega \boldsymbol{\sigma}\,\text{grad}\,N_\alpha\,\mathrm{d}\Omega + \frac{1}{M_\alpha}\int_{\Gamma_h} \boldsymbol{h}N_\alpha\,\mathrm{d}\Gamma \qquad \text{no summation on} \quad \alpha \tag{2}$$

where Ω is the domain of the body; Γ_h is the natural boundary with traction $\boldsymbol{h}$ specified; ρ is the density; $\beta = 1, N^E$; and M_α is the lumped mass at node α.

B. FE-Based Motion Points

If the elements used in the FEM are all constant-strain triangles and a one-point quadrature rule is used to perform the integrations, Eq. (2) becomes

$$\dot{\boldsymbol{v}}_\alpha = -\frac{1}{M_\alpha}\sum_{j=1}^{N_j} A_{[j]}\boldsymbol{\sigma}_{[j]}\,\text{grad}\,N_{\alpha:[j]} + \frac{1}{M_\alpha}\sum_{k=1}^{N_h} L_{[k]}\boldsymbol{h}_{[k]}N^E_{\alpha:[k]} \tag{3}$$

where N_j is the number of elements that have the node α as one of their nodes; N_h is the number of elements that have at least one edge on the boundary and have the node α as one of their node; $A_{[j]}$ is the area of element j; $\sigma_{[j]}$ is the stress at the integration point of element j; $N_{\alpha:[j]}$ is the shape function of node α in element j; $L_{[k]}$ is the length of the element edge k; and $\boldsymbol{h}_{[k]}$ and $N^E_{\alpha:[k]}$ are the traction and the shape function N_α evaluated at center of the edge k, respectively.

Equation (3) can be used in step 1 of the HPM formulation to compute accelerations at motion points on the boundaries. Figure 1c illustrates the use of such a motion point on a stress-free surface. Stress-free corners, surfaces with applied tractions, and the interior of a domain can all be similarly treated. By viewing pseudocells in the HPM setup as elements, the terms related to element shape in Eq. (3), that is, the areas and shape functions, can be computed based on the deformed shape of the corresponding pseudocells. The stress at the integration point of a certain element is provided by the stress at the stress point of the corresponding pseudocell. This is possible because the stress point is located at the centroid of the pseudocell, the same location where the integration point of the corresponding constant strain element is located.

IV. Stability of FE-Based Motion Points

For transient problems, the stability of explicit methods is a major concern for both the HPM and FEM.[5] The FEM is based on a weak formulation. Using eigenvalue analysis, the stability analysis of the explicit FEM leads to an estimation of the maximum time step Δt for the stability of individual elements. For particle methods that are based on the strong form of the momentum conservation equation, "windows" of stability[6] exist when the method 1) employs both motion and stress points and 2) uses MLS to find the stress derivatives and velocity gradients. Different windows of stability for different motion points prevent the use of a uniform time-step size for the entire model.

In HPM, we have found that parametrically locating the stress points within pseudotriangular cells in two dimensions and tetrahedrons in three dimensions exhibits good stability for deformations that do not entail a change of neighborhoods for particles. This allows rather simple stability calculations to be performed by employing a time-step calculation method (step 5 of the HPM formulation) based upon distance $h_{\min}$ and, as important, the use of a single Δt_{crit} for the entire model. Once the deformation levels require the evolution of at least some of the neighborhoods, then the stress points in those areas of the model must be "unlocked" and tracked explicitly. More complex stability calculations must then be performed, and the predictor-corrector methods[6] might be needed to enlarge the windows of stability.

For deformation levels in which neighborhoods are not evolving, the stable critical time steps for an HPM analysis are smaller than those calculated in the pure FEM when the subcells in the HPM analysis are viewed as elements in the FEM analysis. The HPM uses the minimum distance between motion points and their stress-point neighbors to calculate the time step, whereas the FEM uses the minimum length of element edges to calculate the time step. The length of an element edge can also be viewed as the distance between motion points and their motion-point neighbors. Apparently, the distance between motion points and their stress-point neighbors is shorter than the distance between motion points and their motion-point neighbors. As a result, incorporating FE-based motion points does not compromise the stability of the HPM. The distance from FE-based motion points to their respective neighboring

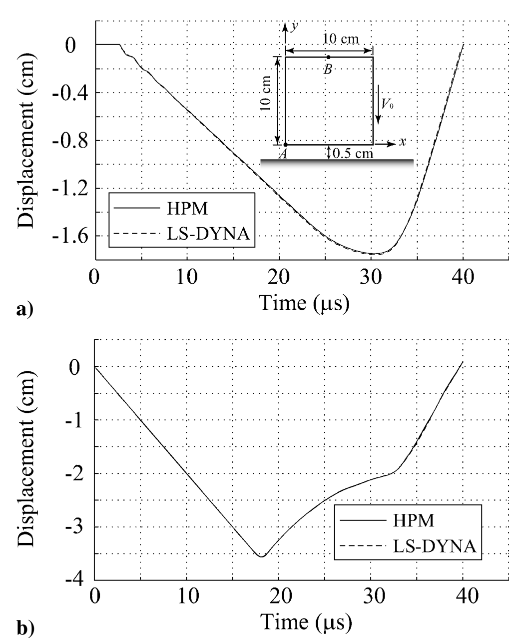

Fig. 2 Comparison of displacement histories for two-dimensional plane strain elastic impact: a) horizontal displacement history of corner point *A* and b) vertical displacement of point *B*. HPM results are obtained where all motion points on the boundary are FE based.

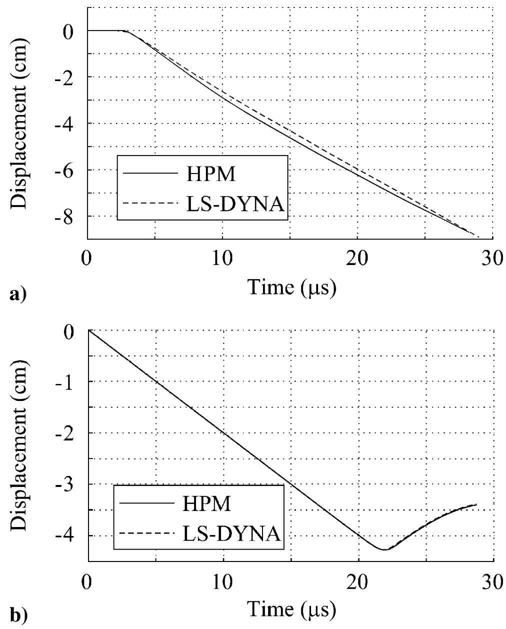

Fig. 3 Comparison of displacement histories for two-dimensional plane-strain inelastic impact: a) horizontal displacement history of corner point *A* and b) vertical displacement of point *B*. HPM results are obtained where all motion points on the boundary are FE based.

stress points will no longer be needed in calculating the time-step length.

V. Numerical Examples

The complete stress-free surface treatment for curved surfaces in HPM is not available yet. This limitation can be removed by employing FE-based motion points on the boundaries. In the present study, only the stress-point neighbors are used in calculating the stress derivatives. Also, Neumann–Richtmyer viscosity formulation[2] is used with linear and quadratic viscosity coefficients of 0.05 and 1.0, respectively. The contact algorithm used for these analyses has been presented earlier.[2]

A body under plane-strain conditions and coming in contact with a flat rigid wall as shown in Fig. 2 is considered. The body travels toward the rigid wall with a speed of 2×10^5 cm/s. The body is discretized using 40×40 rectangular cells and a triangular pseudocell arrangement. The stress points are located at the centroids of the subcells and are tracked parametrically. All motion points on the boundary are treated using the FE-based methodology presented in this Note.

The body is first assumed to be made of elastic material with an elastic modulus of 2×10^7 N/cm^2 and Poisson's ratio of 0.3. Numerical solutions to this problem are obtained using the HPM formulation and the commercial explicit FE code LS-DYNA, respectively. A comparison of the displacement histories is shown in Figs. 2a and 2b, respectively. The body is next assumed to be made of inelastic material with linear isotropic work hardening. The tangential modulus and the yield stress are chosen to be 6.67×10^4 and 6.803×10^4 N/cm^2, respectively. Comparisons of displacement histories are shown in Figs. 3a and 3b, respectively. A close agreement between the results of the HPM and the explicit FEM for both elastic and inelastic impact is observed.

VI. Summary

Finite element (FE)-based motion points are developed for use in particle methods. These motion points are advantageous for boundary treatment in the hybrid particle methods. The resulting approach considers boundary motion points as finite element nodes, whereas the neighboring stress points are considered as the integration points of an explicit finite element method (FEM) formulation. The stability of employing FE-based motion points is discussed. Numerical results are presented to demonstrate the feasibility, accuracy, and stability of the present development. These developments can facilitate methodologies to combine the hybrid particle method and the explicit FEM in a single analysis.

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